2018 Fall
CTP431: Music and Audio Computing

Sound Representations

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Outlines

- Introduction

- Time-domain representation
  - Waveform

- Frequency domain representation
  - Discrete Fourier Transform (DFT)

- Time-Frequency domain representation
  - Short-time Fourier Transform (STFT)
  - Spectrogram

Introduction

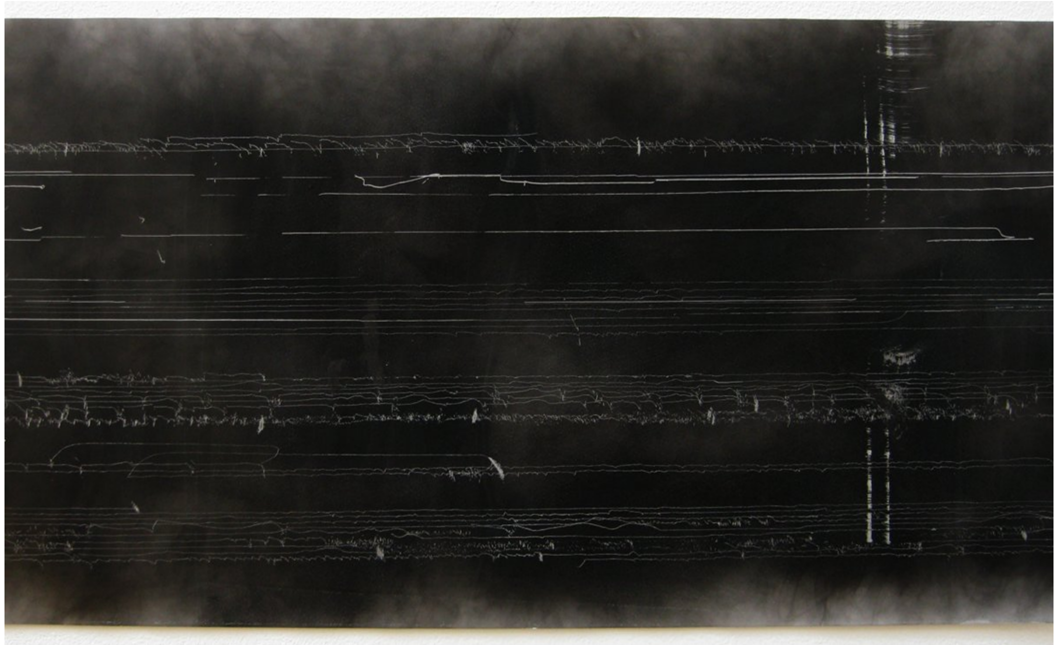
- Visualizing sound as image or animation is very important
  - For research purpose
    - Analyzing the properties of sound: loudness, pitch and timbre
    - More complicated patterns in different contexts
  - For artistic purpose
    - Mapping the sound properties to visual elements
    - Visual elements become more important in music

- In this topic, we will focus on visualizing sound “as it is”

Time-domain Representation

- The raw waveform: the amplitude of sound over time

- Phonautograph (Leon Scott,1857)
  - The first invention of sound recording
  - Recent research on image to sound restoration: http://firstsounds.org/



Source: http://edcarter.net/home/phonautogram/

Time-domain Representation

- Zoom-In view
  - Loudness: yes
  - Pitch: yes if the waveform is periodic (monophonic)
  - Timbre: to some extent from the wave shape (e.g. round or squared)

- Zoom-out view
  - Loudness: yes
  - Pitch: no
  - Timbre: to some extent from the amplitude envelop

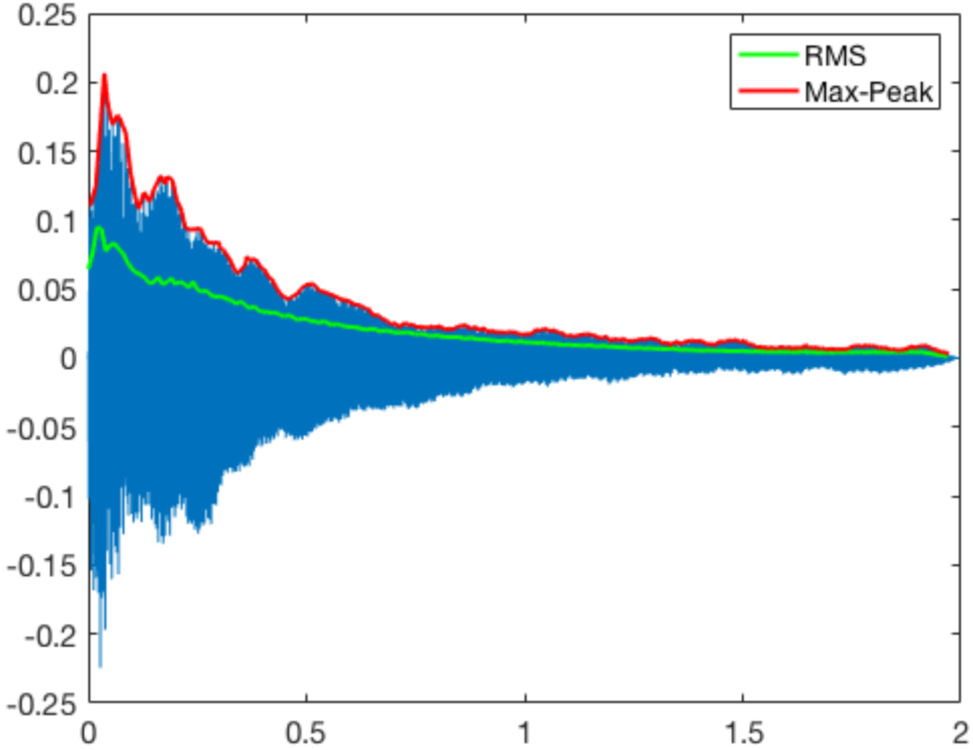
Amplitude Envelope

- Summarized visualization of the waveform
  - Computed by max-peak picking or root-mean-square (RMS)

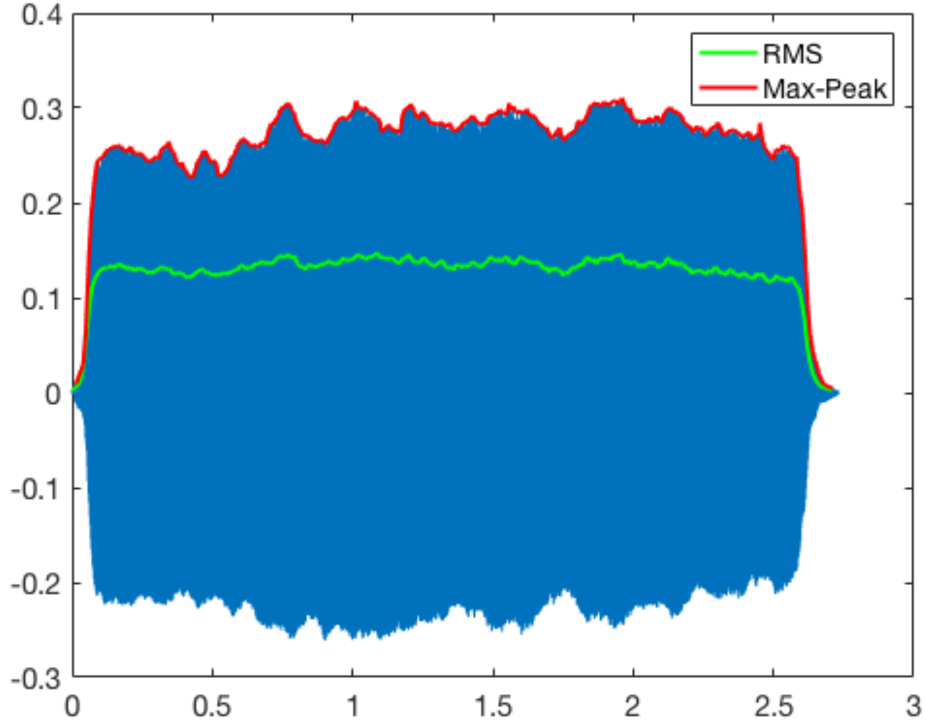
- Parameterized with “ADSR” for musical tones
  - Attack time, Decay time, Sustain level and Release time

- Used to determine gain in dynamic range compression:
  - e.g. compressor, expander

Example: Amplitude Envelope



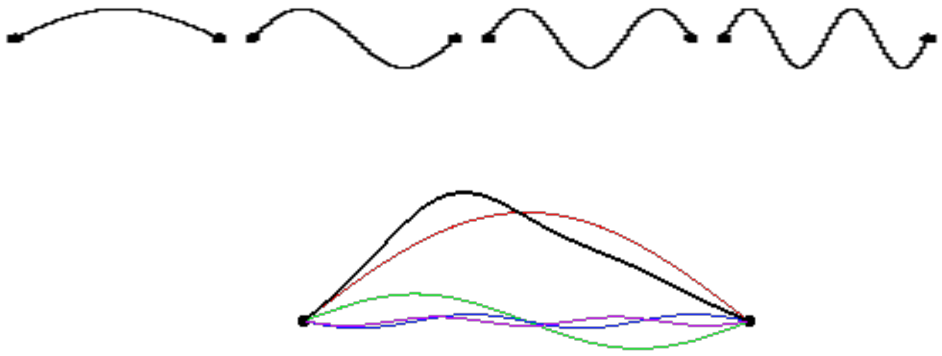
Piano C4 Note



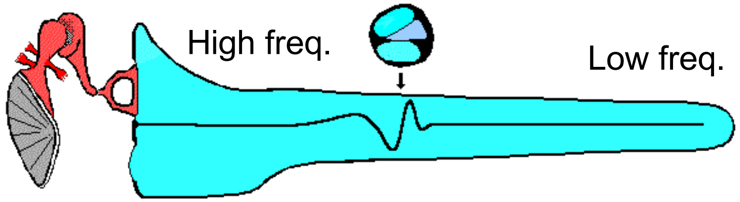
Flute A4 Note

Tone Generation and Perception Perspective

- Musical tones are generated as a combination of (sinusoidal) oscillation modes
- Cochlear has frequency-selective responses



Modes



Cochlear

Source: https://www.acs.psu.edu/drussell/Demos/string/Fixed.html

Source: http://acousticslab.org/psychoacoustics/PMFiles/Module03a.htm

Frequency-Domain Representation

- Can we represent $x(n)$ with a finite set of sinusoids?
  - $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} A(k) r_k(n)$
    - $r_k(n) = \cos(\frac{2\pi kn}{N} + \phi(k))$: discrete-time sinusoid with length *N*
  - Find $A(k), \phi(k)$

Euler's identity

- Euler's identity

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

  - Can be proved by Taylor's series
  - If $\theta = \pi$, $e^{j\pi} + 1 = 0$ ("the most beautiful equation in math")

- Properties

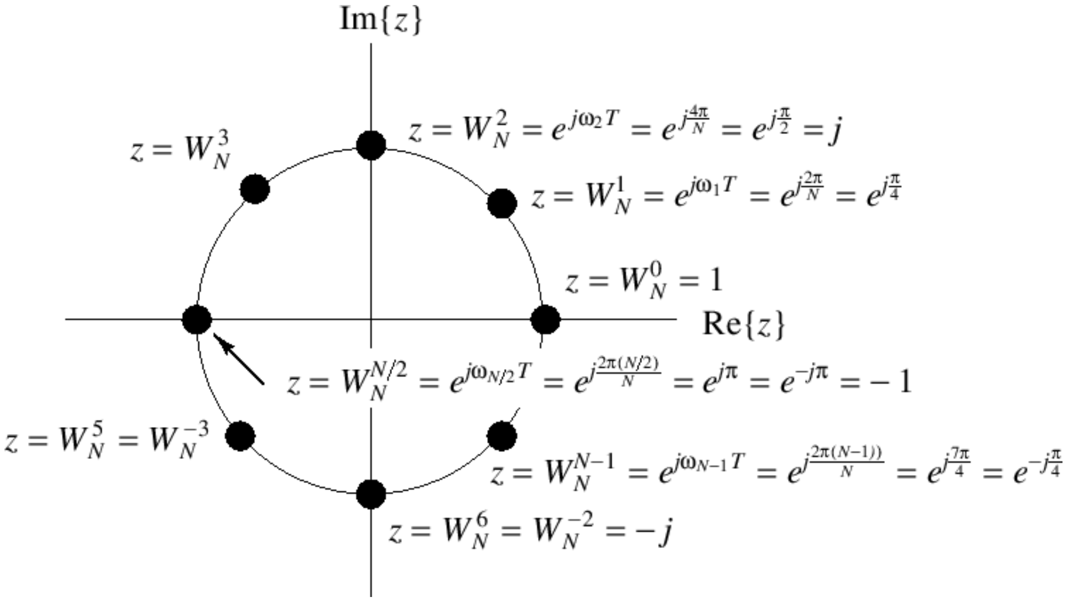
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex Sinusoids

- Cosine and sine can be represented in a single term

$$\boxed{s_k(n) = e^{j\frac{2\pi kn}{N}} = \cos\frac{2\pi kn}{N} + j\sin\frac{2\pi kn}{N}}$$

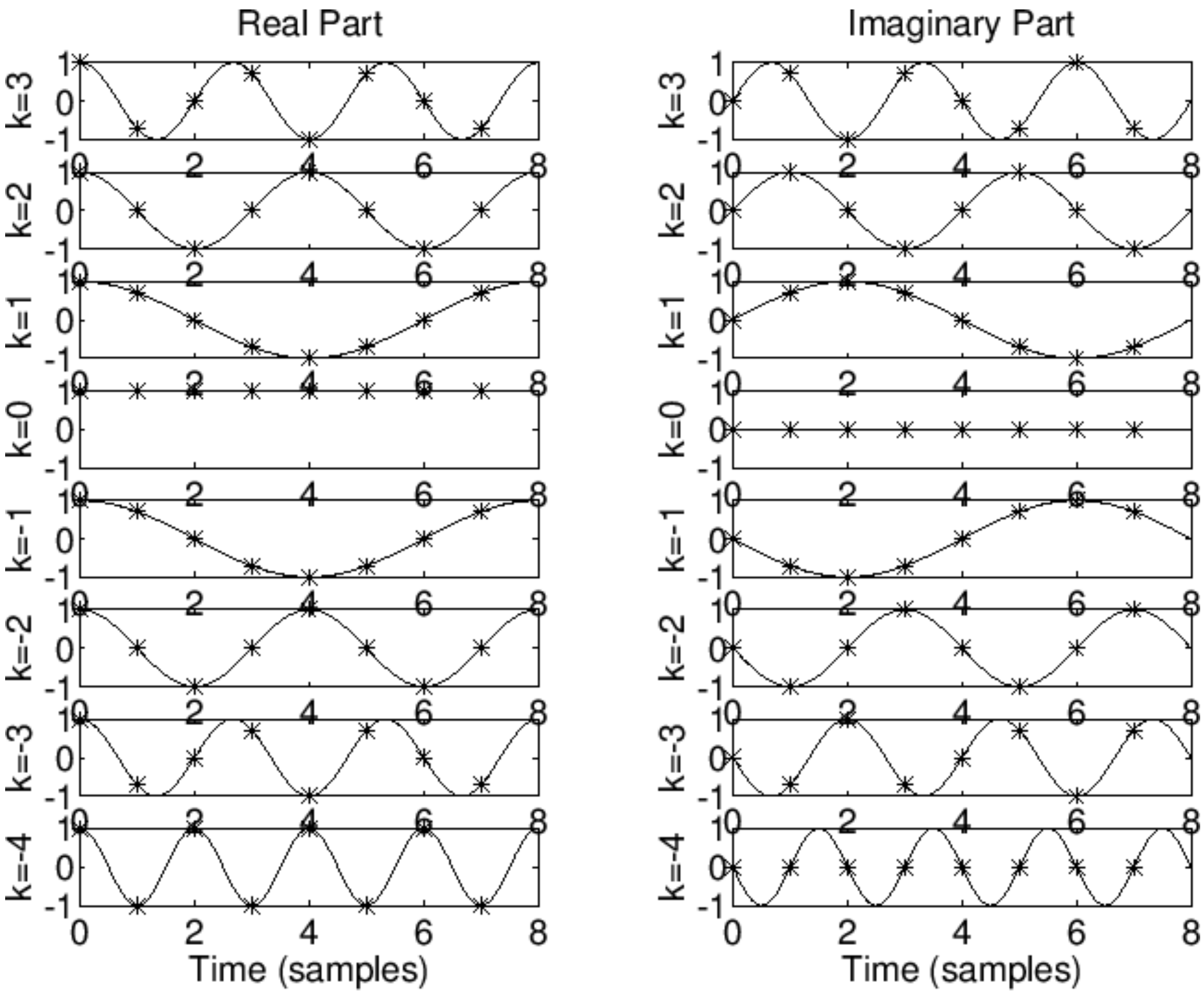
  - Frequencies: $\frac{2\pi k}{N}$ radian or $\frac{k}{N}F_s$ *Hz* (F_s: the sampling rate) ($K = 0, 1, 2, \ldots, N-1$)
  - Example: N = 8



Figures are from https://ccrma.stanford.edu/~jos/dft/

Complex Sinusoids

N = 8



Figures are from https://ccrma.stanford.edu/~jos/dft/

Frequency-Domain Representation Using Complex Sinusoids

- $x(n)$ is expressed in a simpler form:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} A(k)\cos\left(\frac{2\pi kn}{N} + \phi(k)\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} A(k)(e^{j(\frac{2\pi kn}{N}+\phi(k))} + e^{-j(\frac{2\pi kn}{N}+\phi(k))})/2 = \frac{1}{N}\sum_{k=0}^{N-1}(A(k)e^{j\phi(k)}e^{j\frac{2\pi kn}{N}} + A(k)e^{-j\phi(k)}e^{-j\frac{2\pi kn}{N}})/2$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}(X(k)e^{j\frac{2\pi kn}{N}} + \overline{X(k)}e^{-j\frac{2\pi kn}{N}})/2 = \mathrm{Real}\{\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}}\}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}}$$

$$\boxed{X(k) = A(k)e^{j\phi(k)} = A(k)(\cos\phi(k) + j\sin\phi(k))}$$

- Now, how can we find $X(k)$?

Orthogonality of Sinusoids

- Inner product between two complex sinusoids

$$s_p(n) \cdot s_q^*(n) = \sum_{n=0}^{N-1} e^{j\frac{2\pi pn}{N}} \cdot e^{-j\frac{2\pi qn}{N}} = \begin{cases} N & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=0}^{N-1} \sin(2\pi pn/N)\sin(2\pi qn/N)) = \begin{cases} 0 & \text{otherwise} \\ N/2 & if\ \ p = q \\ -N/2 & if\ \ p = N-q \end{cases}$$

$$\sum_{n=0}^{N-1} \cos(2\pi pn/N)\sin(2\pi qn/N)) = 0$$

$$\sum_{n=0}^{N-1} \cos(2\pi pn/N)\cos(2\pi qn/N)) = \begin{cases} N/2 & if\ \ p = q \text{ or } p = N-q \\ 0 & \text{otherwise} \end{cases}$$

Orthogonal Projection on Complex Sinusoids

- Do the inner product with the signal and sinusoids

$$x(n) \cdot s_k(n) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} (\frac{1}{N} \sum_{l=0}^{N-1} X(k) e^{j\frac{2\pi ln}{N}}) e^{-j\frac{2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X(k) (\sum_{n=0}^{N-1} e^{j\frac{2\pi ln}{N}} e^{-j\frac{2\pi kn}{N}}) = \frac{1}{N} X(k) N = X(k) = A(k) e^{j\phi(k)}$$

To Wrap Up

- Discrete Fourier Transform

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}} = X_R(k) + jX_I(k) = A(k)^{j\phi(k)}$$

  - Magnitude spectrum: $|X(k)| = A(k) = \sqrt{X_R^2(k) + X_I^2(k)}$
  - Phase spectrum: $\angle X(k) = \phi(k) = \tan^{-1}(\frac{X_I(k)}{X_R(k)})$

- Inverse Discrete Fourier Transform

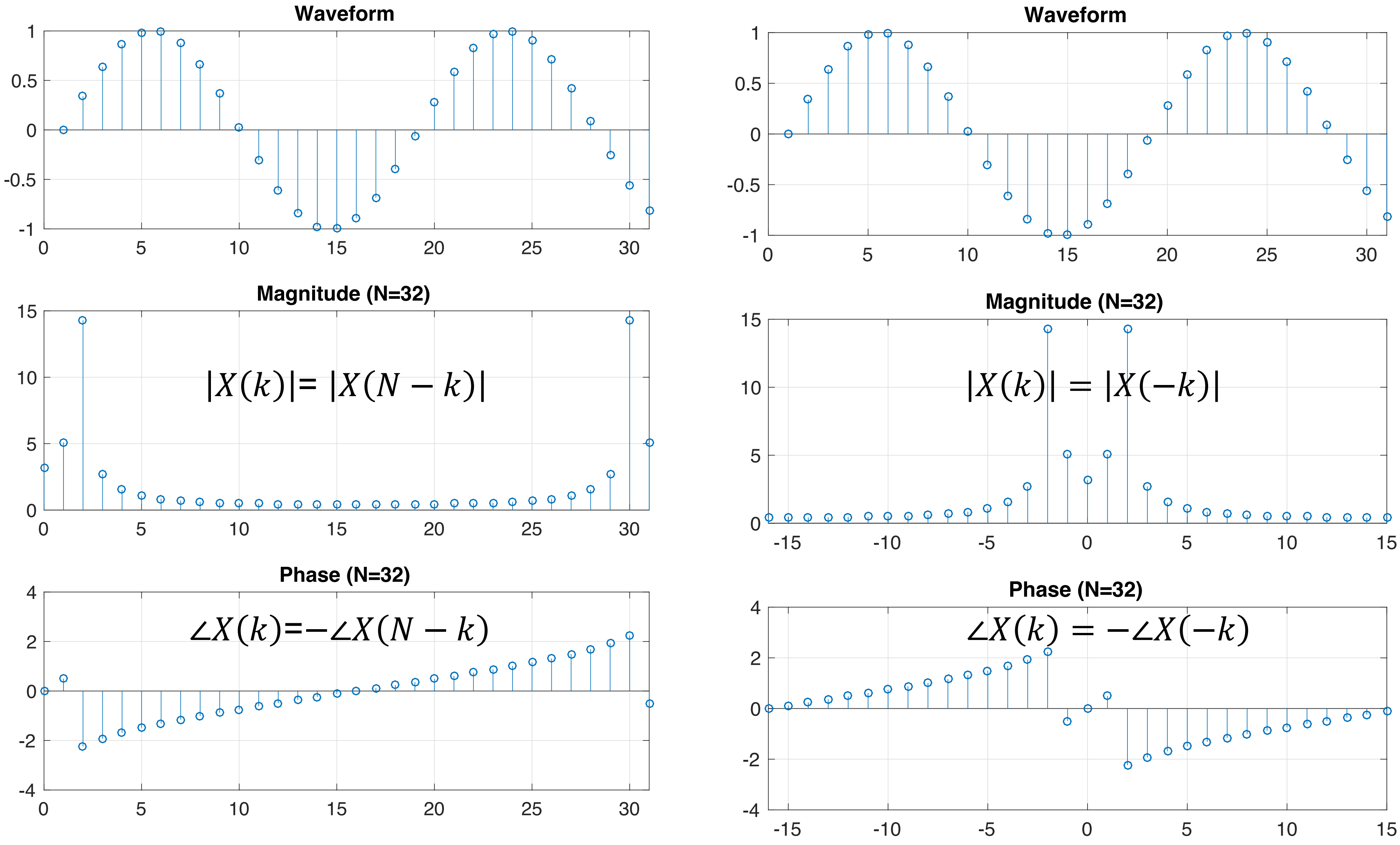
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi kn}{N}}$$

Properties of DFT

- Periodicity
  - $X(k) = X(k+N) = X(k+2N) =$ ...
  - $X(k) = X(k-N) = X(k-2N) =$ ...

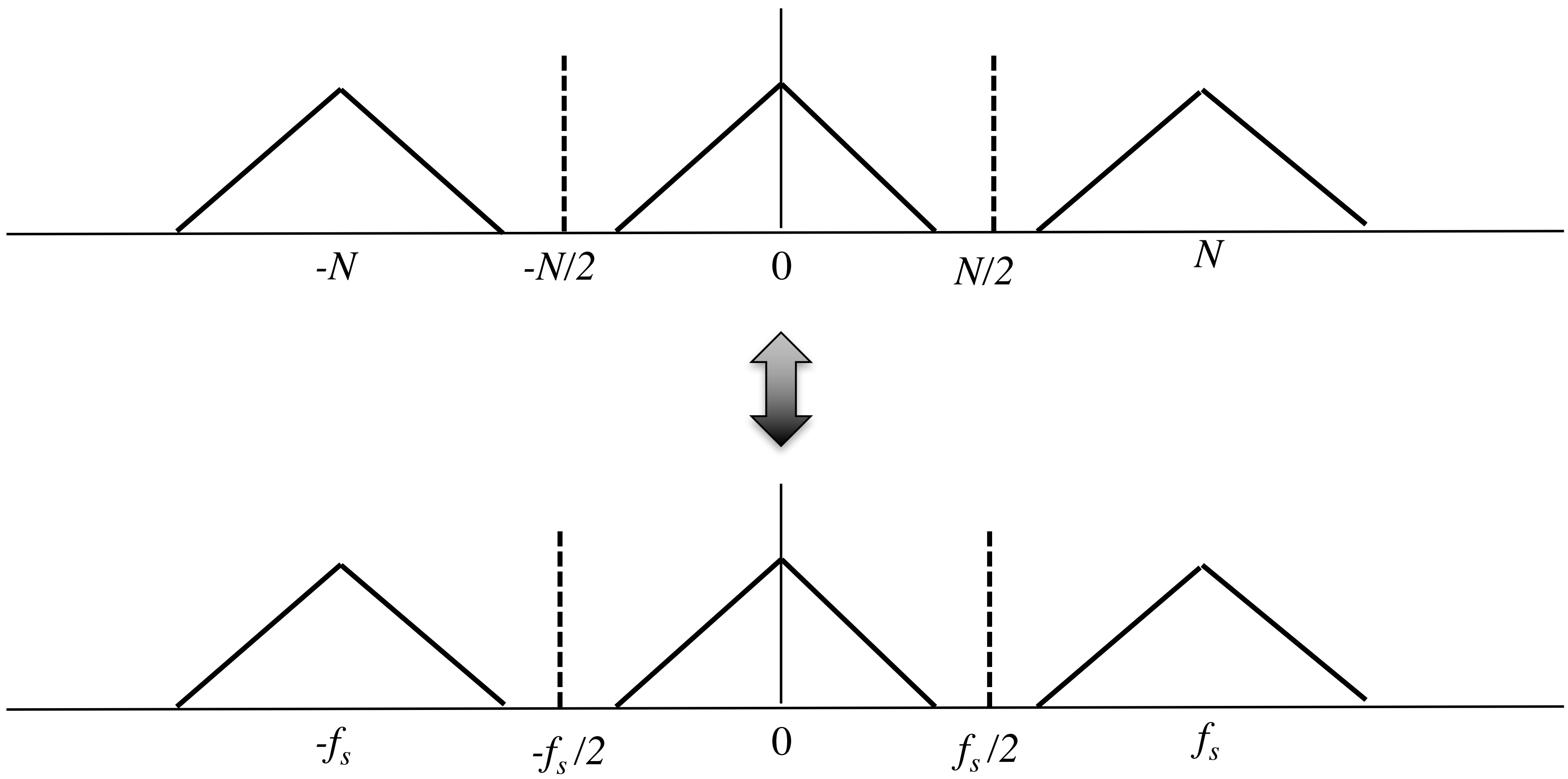
- Symmetry
  - Magnitude response: $|X(k)| = |X(-k)|$ = $|X(N-k)|$
  - Phase response : $\angle X(k) = -\angle X(-k)$ =$-\angle X(N-k)$
  - We often display only half the amplitude and phase responses

Properties of DFT

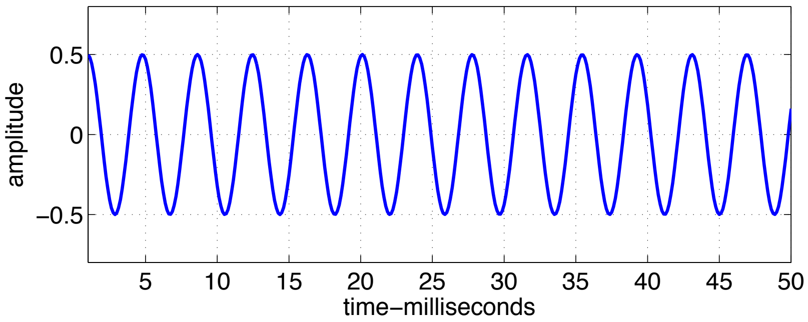


Frequency Scaling

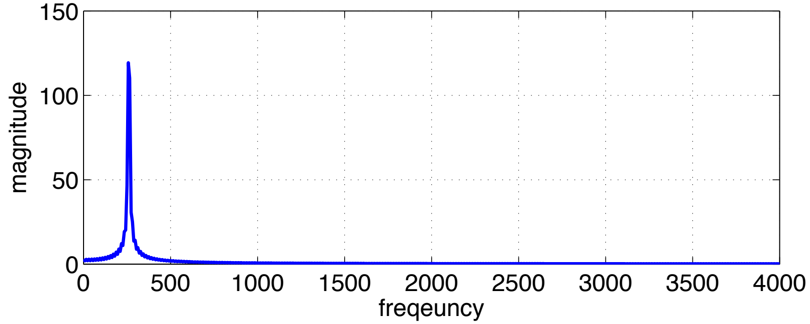
- $X(k)(k = 0, 1, \dots, N)$ corresponds to frequency values that are evenly distributed between 0 and fs in Hz



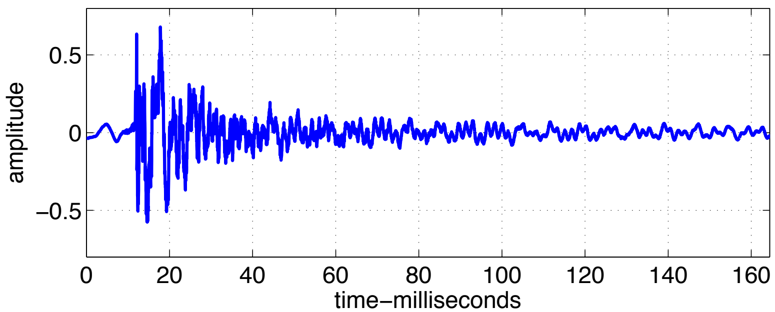
Examples of DFT



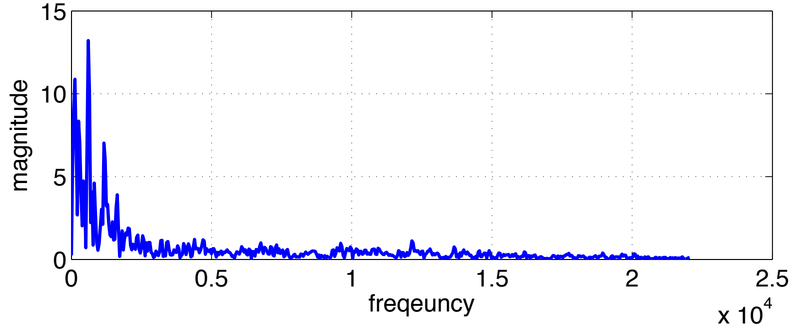
Sine: waveform



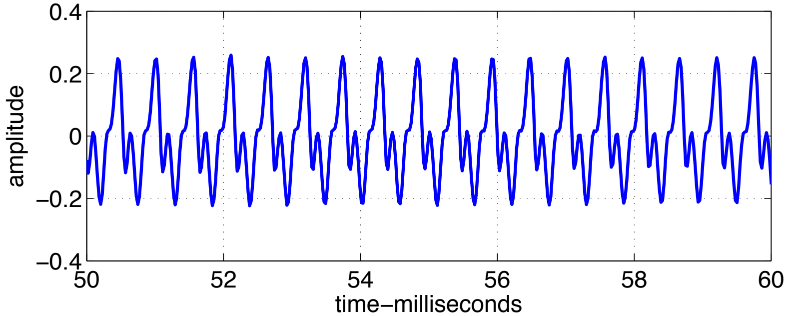
Sine: spectrum



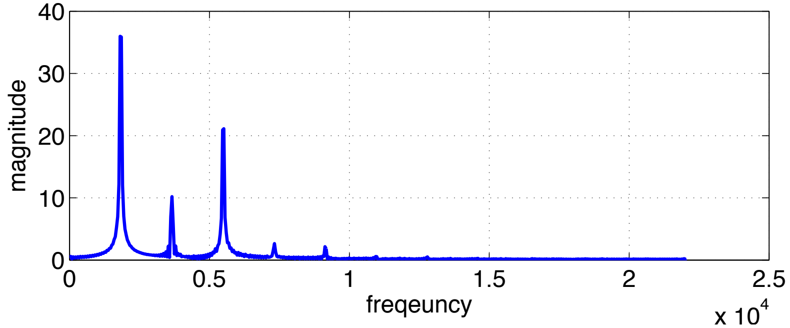
Drum: waveform



Drum: spectrum



Flute: waveform



Flute: spectrum

Fast Fourier Transform (FFT)

- Matrix multiplication view of DFT

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ \vdots \\ X(N-2) \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ 1 & W_N^3 & W_N^6 & \cdots & W_N^{3(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \\ & & & \cdots & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix}$$

- In fact, we don't compute this directly. There is a more efficiently way, which is called "Fast Fourier Transform (FFT)"
  - Complexity reduction by FFT: O(N^2)→ O($N\log_2 N$)
  - Divide and conquer

Time-Frequency Domain Representation

- DFT assumes that the signal is stationary
  - It is not a good idea to apply DFT to a long and dynamically changing signal like music
  - Instead, we segment the signal and apply DFT separately

- Short-Time Fourier Transform

$$X(k,l) = \sum_{n=0}^{N-1} w(n)x(n + l \cdot h)e^{-j\left(\frac{2\pi kn}{N}\right)}$$

h : hop size

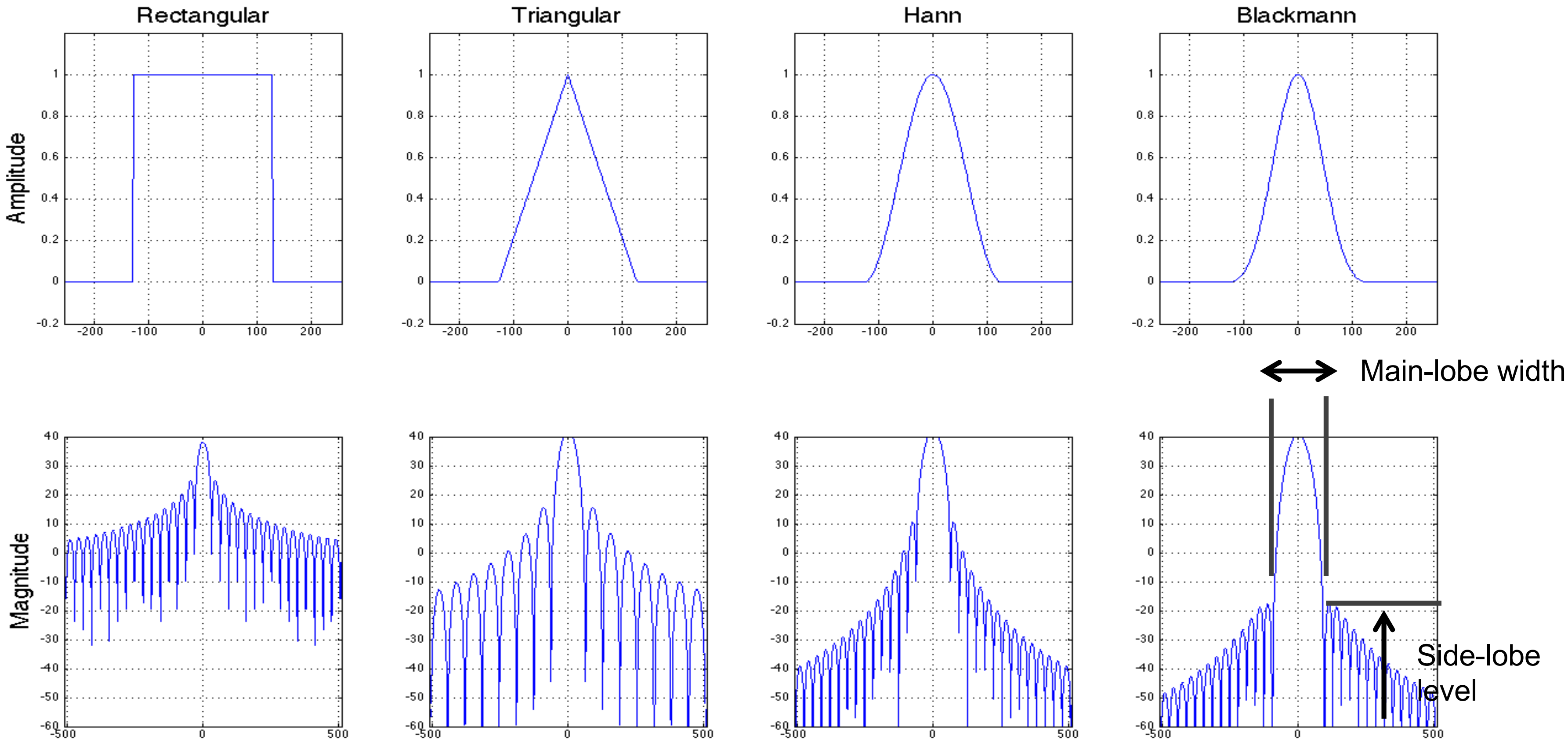
$w(n)$: window

N : FFT size

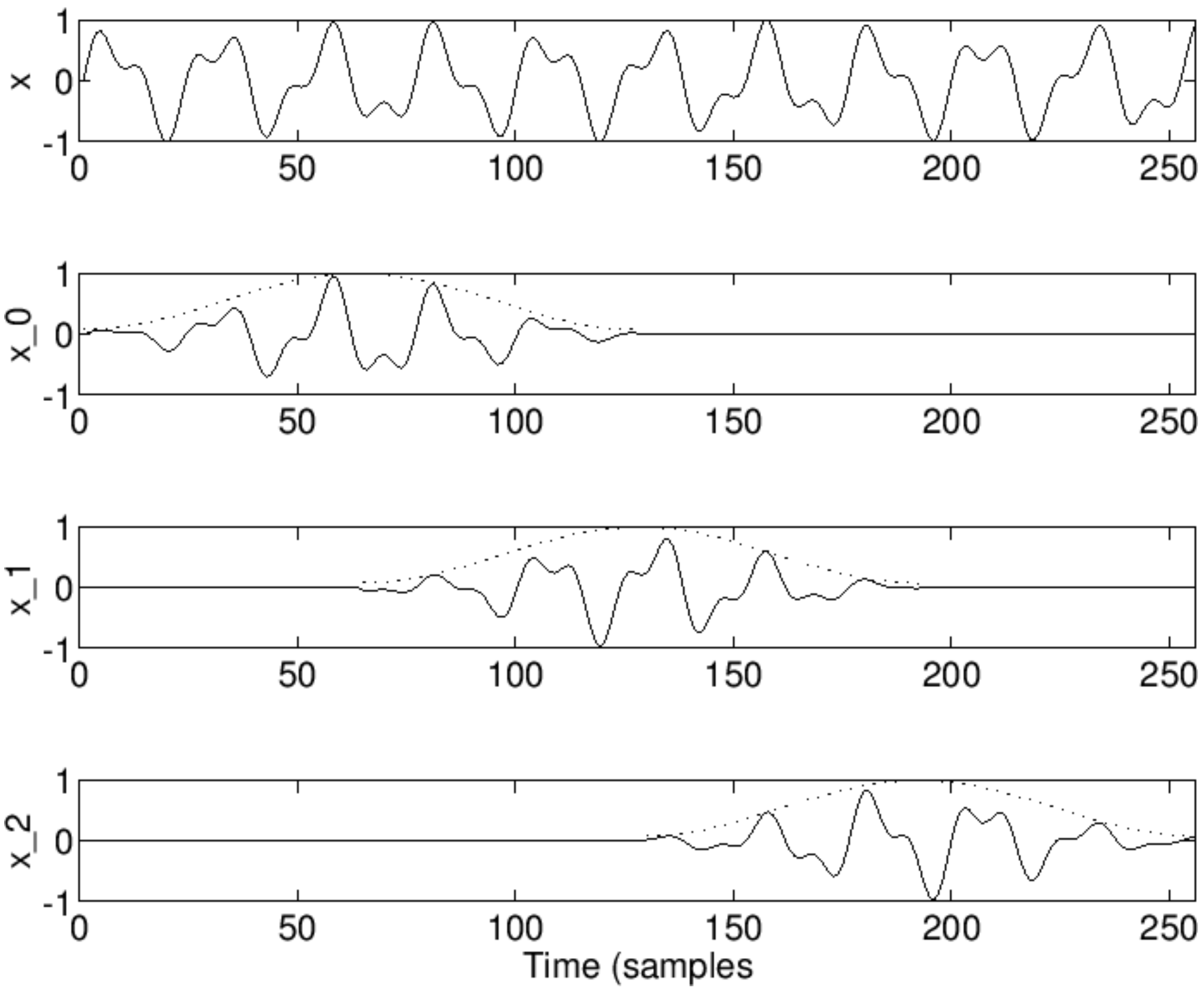
- This produces 2-D time-frequency representations
  - Parameters: window size, window type, FFT size, hop size
  - "Spectrogram" from the magnitude

Windowing

- Types of window functions
  - Trade-off between the width of main-lobe and the level of side-lobe

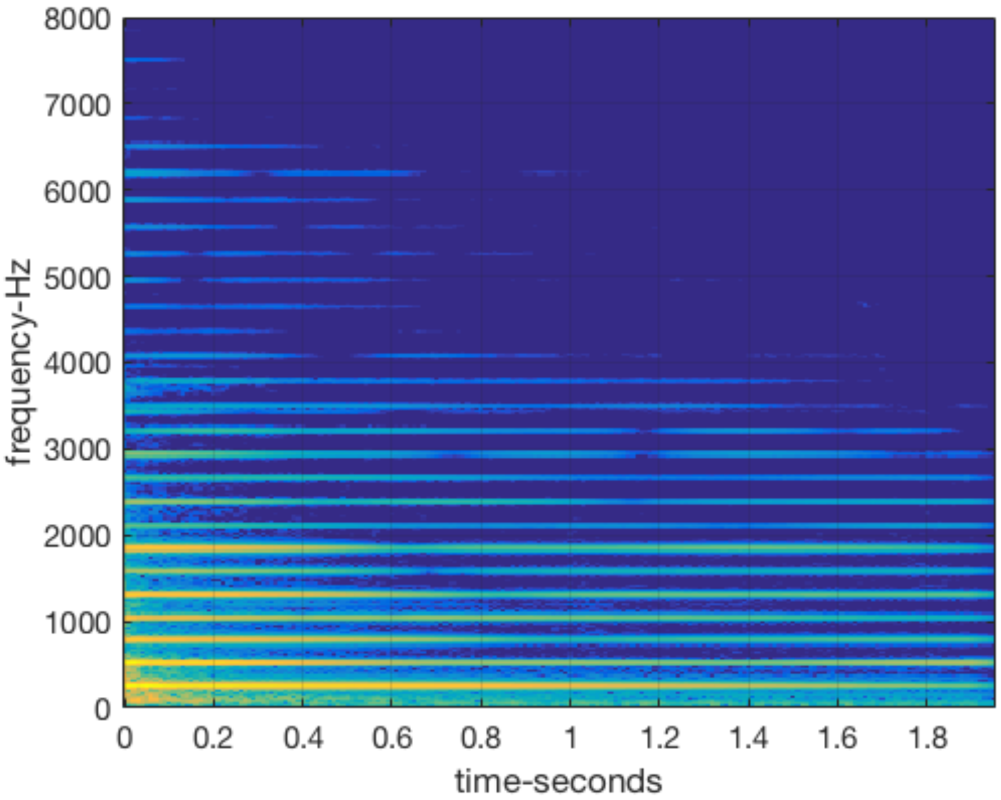


Short-Time Fourier Transform (STFT)

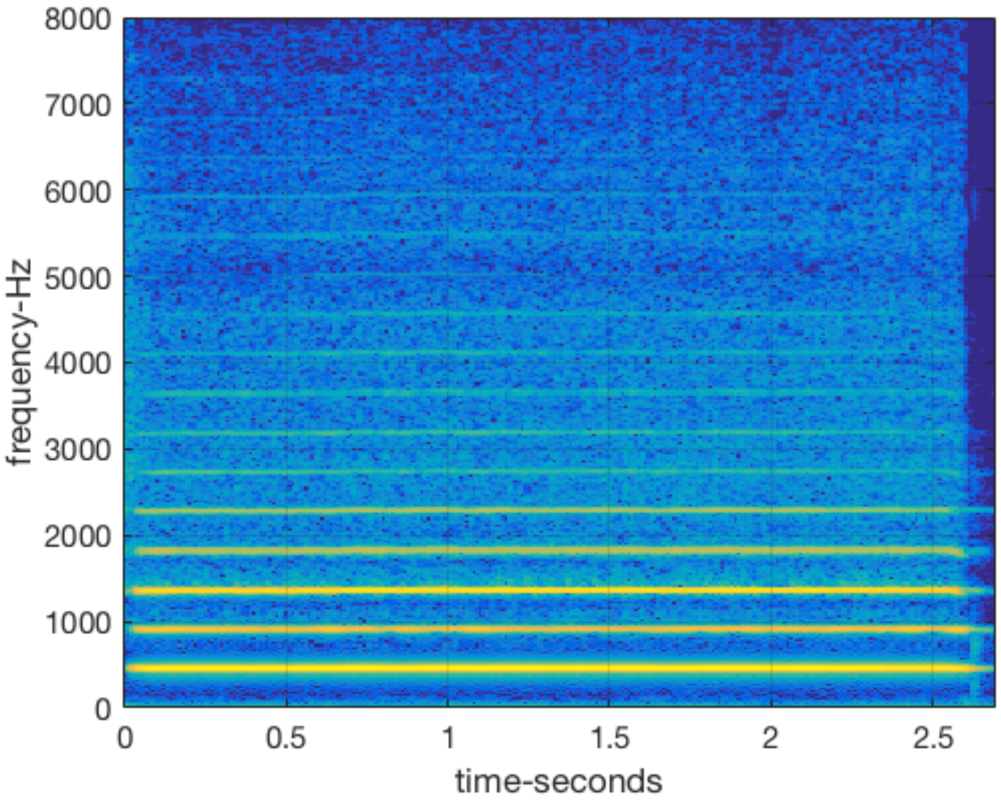


50% overlap

Example: Spectrogram

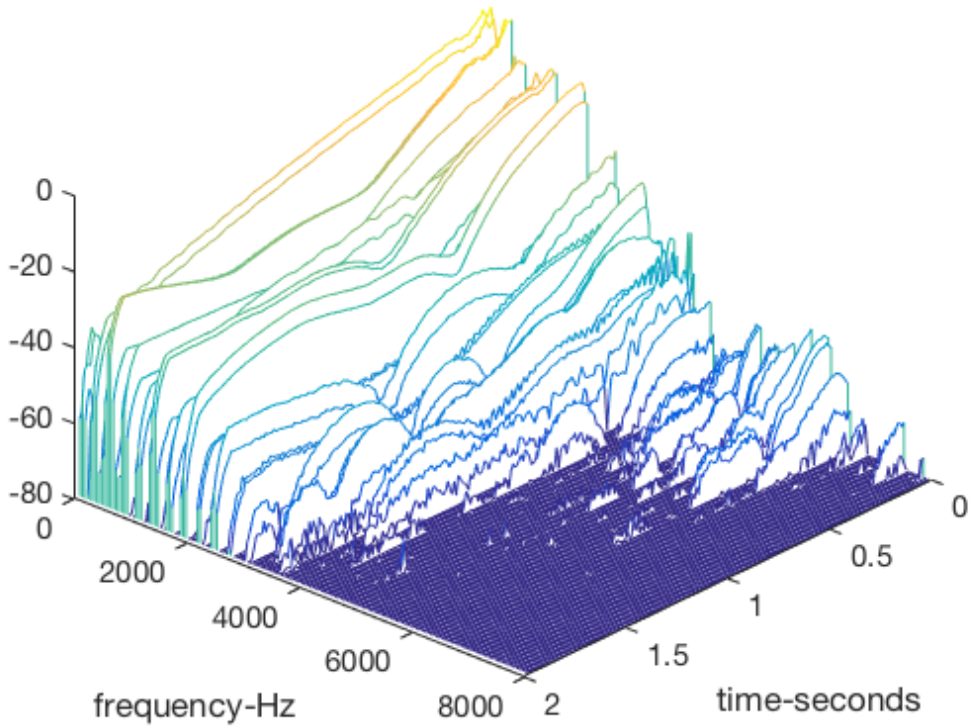


Piano C4 Note

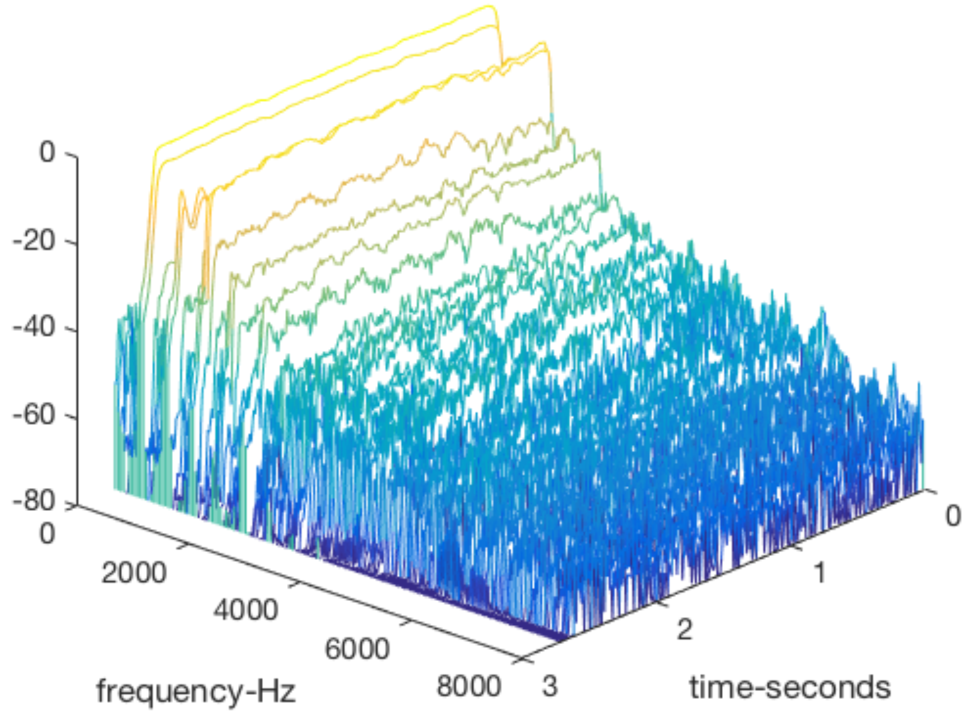


Flute A4 Note

Example: Spectrogram - 3D waterfall

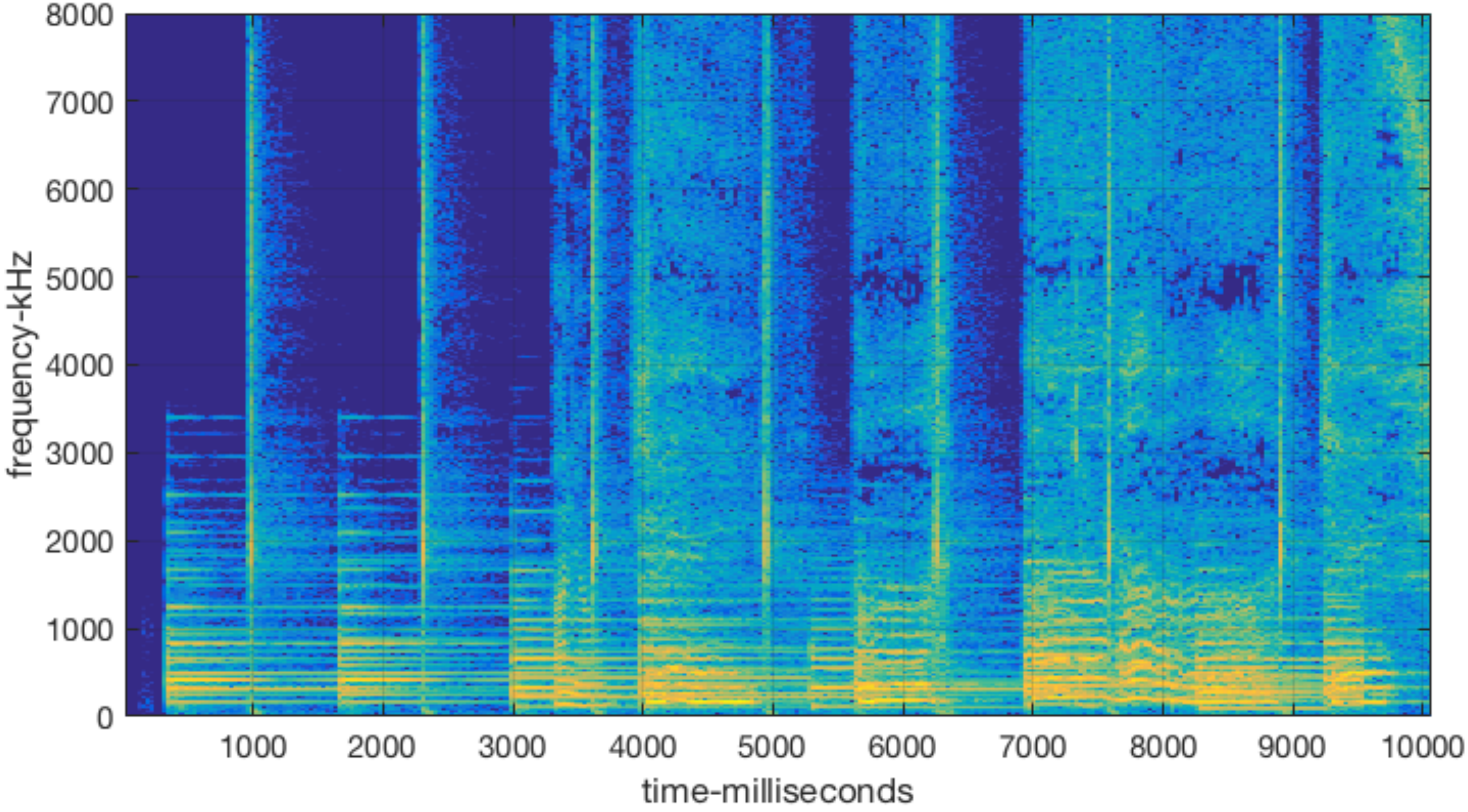


Piano C4 Note

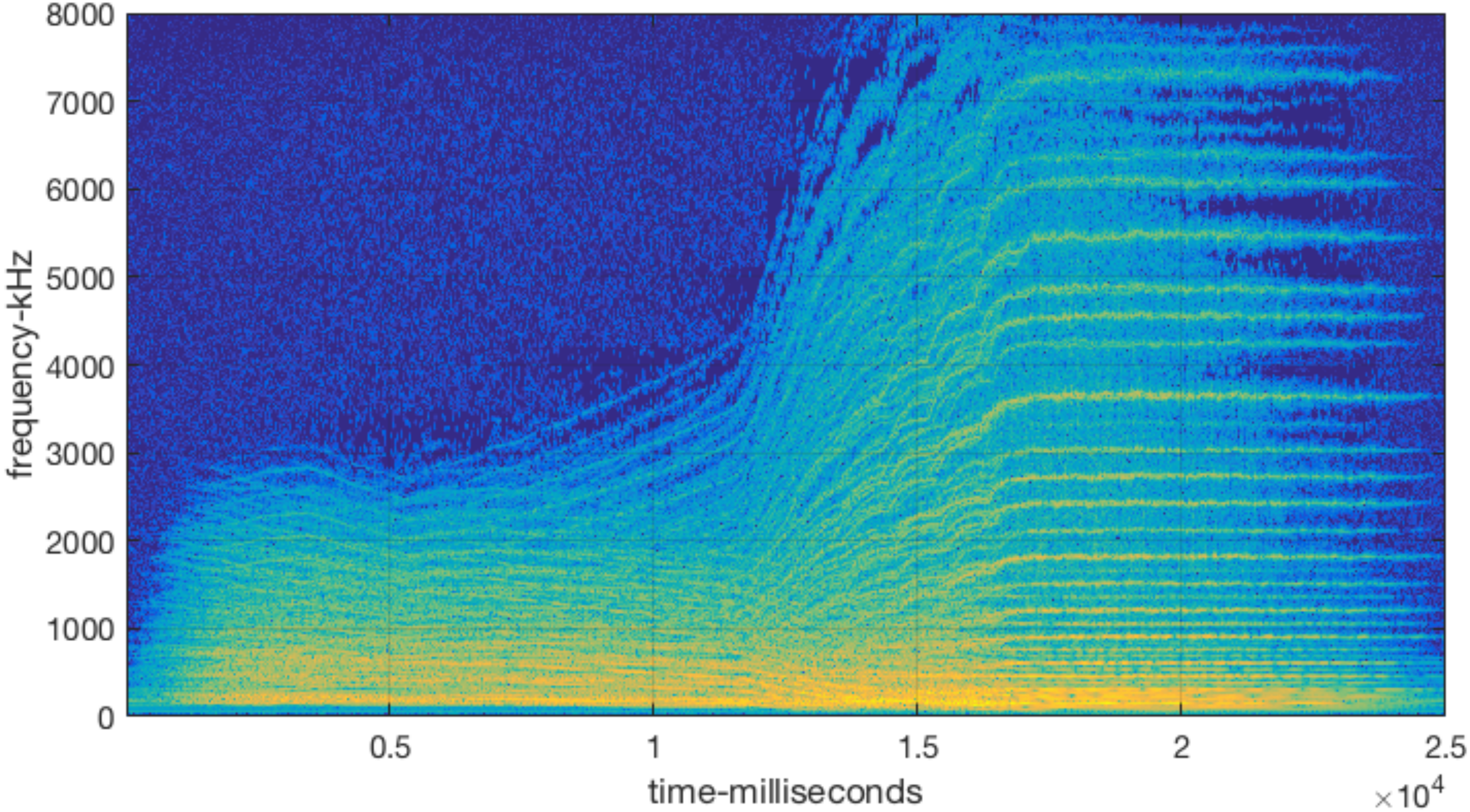


Flute A4 Note

Example: Pop Music

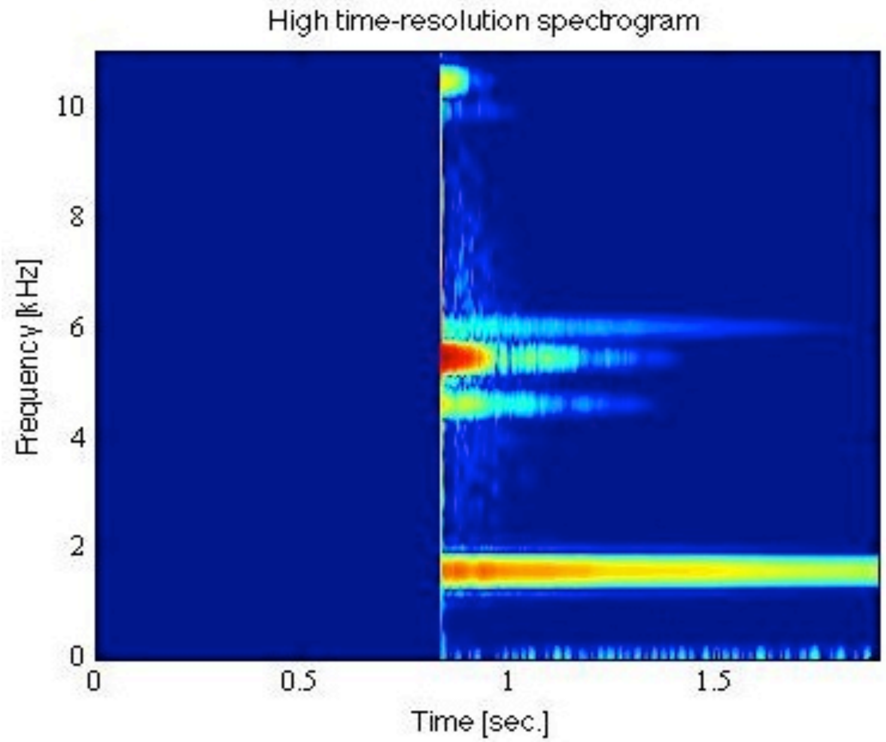


Example: Deep Note

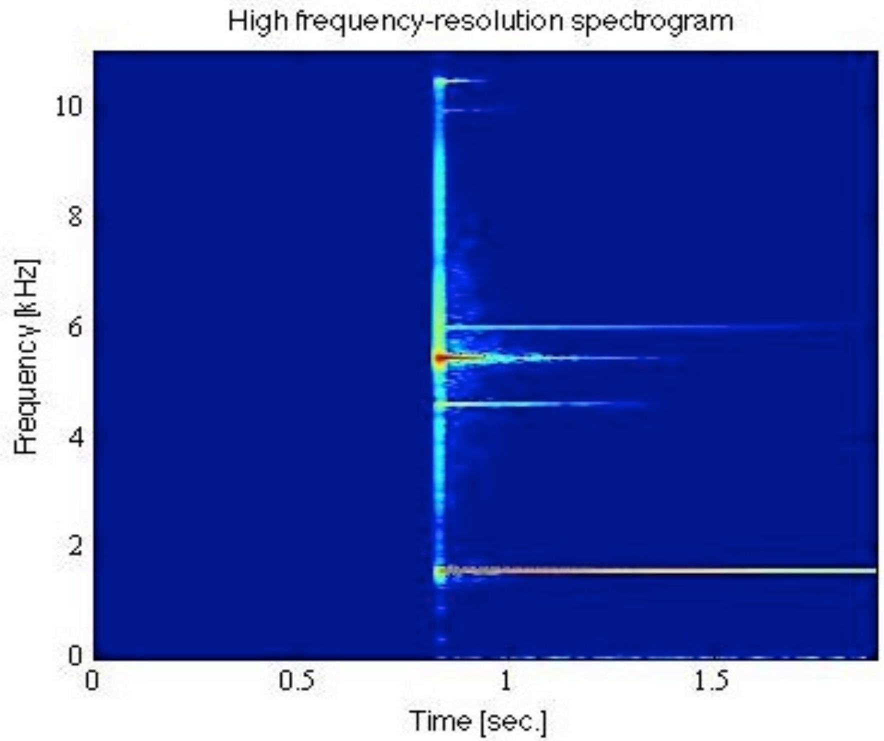


Time-Frequency Resolutions in STFT

- Trade-off between time and frequency resolution by window size



Short window
High time resolution
Low freq. resolution



Long window
High freq. resolution
Low time resolution